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A Symbolic Demonstration of Hilbert's Method for deriving Invariants and Covariants of given Ternary Forms.

By Henry S. White.

In the course of a recent paper, now fast becoming famous, Dr. Hilbert of Koenigsberg has developed a process of derivation by means of substitution which is likely to prove an important contribution to the theory of invariants and covariants.* Strictly speaking, the process itself is not new, but to Dr. Hilbert is due the credit of having adapted it to an entirely new end. Practically the same operation was employed by Clebsch,† Gordan,‡ and F. Mertens‖ in studying the formal constitution of invariants of linear transformation. Hilbert, detecting the rationale of this use, applies the operation to any term or sum of terms that can form part of an invariant expression, and shows that the result is a complete invariant. While the summary proof that he gives is sufficient and elegant, yet it seems possible to simplify both the statement of the operation and the proof of its character. For this purpose I employ the symbolic notation of Aronhold and Clebsch for the coefficients of a ternary form; thereby the invariant character of the result becomes evident upon mere inspection. Hilbert's method then admits of restatement as an aggregate of extremely simple groupings and substitutions; a statement which can then be extended at once, as Hilbert's has been, to quaternary and n-ary quantics.

The derivation of covariants is possible by the same method with slight modification, and the required covariant may contain as many cogredient sets of variables as one wishes. Further, it may be a function of several distinct sets of variables, each set subject to an independent linear transformation; *e. g.* it may be a combinant of several quantics of equal order. I shall give a general statement of the method as applied to ternary covariants, referring for proof to the discussion upon invariants. In conclusion I present, as a useful application, an easy proof of an important (and perhaps not sufficiently well-known) theorem on covariants.

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*Ueber die Theorie der algebraischen Formen, Math. Ann. 36, pp. 524–6.

† Ueber symbolische Darstellung algebraischer Formen, Crelle's Journal, 59, pp. 7–15.

‡ Gordan-Kerschensteiner, Vorlesungen über Invariantentheorie, Bd. 2, pp. 114–119.

‖ Ueber invariante Gebilde ternärer Formen, Sitzungsberichte der Math.-Naturwissensch. Classe der kais. Akad. der Wissenschaften, Wien, 1887, pp. 942–992.

§1.—*Hilbert's Auxiliary Substitution and Cayley's Operator Ω combined and expressed in Clebsch-Aronhold Symbolic define a substitutional Operation giving rise to Invariants.*

In order to define the weights of a function of the coefficients of a ternary form, I follow a common notation in writing each coefficient with three subscript indices:

$$f(x_1,\ x_2,\ x_3)=\sum_{(i+k+l=n)}\frac{n\,!}{i\,!\ k\,!\ l\,!}\cdot a_{ikl}x_1^i x_2^k x_3^l\,. \tag{1}$$

These three indices may be termed the first, second, and third weights respectively of the coefficient a_{ikl}. In any product of coefficients the first weight will mean the sum of the first weights of the several factors, etc.,

$$A_{IKL}=a_{ikl}\cdot a_{i'k'l'}\cdot a_{i''k''l''}\cdot\ldots\ldots(\text{to } g \text{ factors}). \tag{2}$$

$$\left.\begin{array}{ll}\text{First weight} & I=\Sigma(i)=i+i'+i''+\ldots\ldots(\text{to } g \text{ terms}),\\ \text{Second weight} & K=\Sigma(k),\\ \text{Third weight} & L=\Sigma(l)=n\,.\,g=(I+K).\end{array}\right\} \tag{3}$$

Every invariant of degree g in the coefficients of the ternary form $f(x_1,\ x_2,\ x_3)$ will be composed of terms all having the same first weight, the same second weight, and consequently the same third weight; and these three weights must be equal; *i. e.* if the invariant with its weights be designated by J_{IKL}, and its degree by g, we must have

$$I=K=L=\tfrac{1}{3}n\,.\,g\,. \tag{4}$$

The problem proposed is, *from a given sum of terms satisfying these conditions as to degree and weights, to derive by some determinate process a complete invariant.*

As a solution of this problem, the derivation-process adopted by Hilbert is the following, consisting of three distinct steps. Suppose the ternary form $f(x_1x_2x_3)$, (1), to have arisen from the form

$$\bar{f}(y_1, y_2, y_3) = \sum_{i+k+l=n} \frac{n!}{i!\;k!\;l!}\,\bar{a}_{ikl} \cdot y_1^i y_2^k y_3^l \equiv f(x_1, x_2, x_3) \tag{5}$$

by the linear substitution

$$\left\{\begin{array}{l} y_1 = \xi_1 x_1 + \eta_1 x_2 + \zeta_1 x_3 \\ y_2 = \xi_2 x_1 + \eta_2 x_2 + \zeta_2 x_3 \\ y_3 = \xi_3 x_1 + \eta_3 x_2 + \zeta_3 x_3 \end{array}\right\} \tag{6}$$

Then the coefficients a_{ikl} of the form f are linear homogeneous functions of the coefficients $\bar{a}_{ikl}$ of $\bar{f}$, and contain moreover the parameters ξ, η, ζ homogeneously to the order n. Let the given polynomial satisfying the conditions (4) be denoted by

$$(A_{IKL}).$$

The first step is this: *Substitute in the polynomial* (A_{IKL}) *for the coefficients* a_{ikl} *their values expressed in terms of the coefficients* $\bar{a}_{ikl}$ *and the parameters,* ξ's, η's, ζ's. Call the resulting polynomial

$$(\bar{A}_{IKL}).$$

Its indices I, K, L now denote its orders in the ξ's, the η's, and the ζ's respectively. Remembering that each of these is equal to $\frac{1}{3}n \,.\, g$, (4), we come to the second step in the operation: *Free the expression from the nine parameters by applying* $\frac{1}{3}n \,.\, g$ *times in succession the operator* Ω:

$$\Omega = \begin{vmatrix} \dfrac{\partial}{\partial\xi_1} & \dfrac{\partial}{\partial\xi_2} & \dfrac{\partial}{\partial\xi_3} \\ \dfrac{\partial}{\partial\eta_1} & \dfrac{\partial}{\partial\eta_2} & \dfrac{\partial}{\partial\eta_3} \\ \dfrac{\partial}{\partial\zeta_1} & \dfrac{\partial}{\partial\zeta_2} & \dfrac{\partial}{\partial\zeta_3} \end{vmatrix} \tag{7}$$

The resulting expression,

$$\Omega^{\frac{1}{3}ng} \,.\, (\bar{A}_{IKL}) = J(\bar{a}_{ikl}), \tag{8}$$

is a function of the coefficients $\bar{a}_{ikl}$. The third step is the following: *Replace in* $J(\bar{a}_{ikl})$ *every coefficient* $\bar{a}_{i'k'l'}$ *by the corresponding coefficient* $a_{i'k'l'}$ *of the form* $f(x_1, x_2, x_3)$; *the function* $J(a_{ikl})$ *is then an invariant of the latter form.*

The proof of the invariant property in this product of three operations I shall exhibit by analysis of the effect of successive steps, writing each symbolically. First of all as a necessary preliminary I suppose each actual coefficient, a_{ikl} or $\bar{a}_{ikl}$, replaced by a symbolic product $a_1^i a_2^k a_3^l (\equiv b_1^i b_2^k b_3^l \equiv \text{etc.})$ or $\bar{a}_1^i \bar{a}_2^k \bar{a}_3^l$; *i. e.* I write each form in the Clebsch-Aronhold notation, thus:

$$\left.\begin{array}{l} f(x_1, x_2, x_3) \equiv (a_1 x_1 + a_2 x_2 + a_3 x_3)^n = a_x^n = b_x^n = c_x^n, \text{ etc.} \\ \bar{f}(y_1, y_2, y_3) \equiv (\bar{a}_1 y_1 + \bar{a}_2 y_2 + \bar{a}_3 y_3)^n = \bar{a}_y^n = \bar{b}_y^n = \bar{c}_y^n, \text{ etc.} \end{array}\right\} \tag{9}$$

By virtue of the linear relations (6) between the x's and the y's, the symbols a are expressed linearly in the symbols $\bar{a}$.*

$$\left\{\begin{array}{l} a_1 = \bar{a}_1 \xi_1 + \bar{a}_2 \xi_2 + \bar{a}_3 \xi_3 \\ a_2 = \bar{a}_1 \eta_1 + \bar{a}_2 \eta_2 + \bar{a}_3 \eta_3 \\ a_3 = \bar{a}_1 \zeta_1 + \bar{a}_2 \zeta_2 + \bar{a}_3 \zeta_3 \end{array}\right\} \tag{9a}$$

Hence follows, *e. g.* the expression of $f(x_1, x_2, x_3)$ in terms of symbols $\bar{a}$:

$$f(x_1, x_2, x_3) = (x_1 \bar{a}_\xi + x_2 \bar{a}_\eta + x_3 \bar{a}_\zeta)^n.$$

From this one sees that the substitution of their values in terms of $\bar{a}_{ikl}$ and the parameters for the a_{ikl} in the polynomial (A_{IKL}) is effected by the operator $S^g_{\xi\eta\zeta}$:

$$(n\,!)^g \,.\, S^g_{\xi\eta\zeta} = \left\{\begin{array}{c} \left(\bar{a}_\xi \dfrac{\partial}{\partial a_1} + \bar{a}_\eta \dfrac{\partial}{\partial a_2} + \bar{a}_\zeta \dfrac{\partial}{\partial a_3}\right)^n . \\ \left(\bar{b}_\xi \dfrac{\partial}{\partial b_1} + \bar{b}_\eta \dfrac{\partial}{\partial b_2} + \bar{b}_\zeta \dfrac{\partial}{\partial b_3}\right)^n . \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \text{(to } g \text{ factors of } n^{\text{th}} \text{ order).} \end{array}\right\} \tag{10}$$

The second operator, Ω, is already symbolically expressed, (7). The third is simply this:

$$(n\,!)^g R^g(\bar{a}, a) = \left\{\begin{array}{c} \left(a_1 \dfrac{\partial}{\partial \bar{a}_1} + a_2 \dfrac{\partial}{\partial \bar{a}_2} + a_3 \dfrac{\partial}{\partial a_3}\right)^n . \\ \left(b_1 \dfrac{\partial}{\partial \bar{b}_1} + b_2 \dfrac{\partial}{\partial \bar{b}_2} + b_3 \dfrac{\partial}{\partial \bar{b}_3}\right)^n . \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \text{(to } g \text{ factors of } n^{\text{th}} \text{ order).} \end{array}\right\} \tag{11}$$

By the aid of these symbols we may write the final result thus:

$$J(a_{ikl}) = R^g(\bar{a}, a) \,.\, \Omega^{\frac{1}{3}ng} \,.\, S^g_{\xi\eta\zeta} \,.\, (A_{IKL}). \tag{12}$$

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* Gordan, *loc. cit.*, pp. 110 and 114.

The advantage of this distinguishing of operators from function or *operand* is, that we are at liberty to combine $\Omega^{\frac{1}{3}ng}$ with $S^g_{\xi\eta\zeta}$ before applying. The simplest grouping of symbolic terms in the combined operation is to be sought; it appears to be the following. *Let the three constituents in any horizontal row of the three-rowed determinant* Ω *act simultaneously upon a single linear factor of* $S_{\xi\eta\zeta}$. The symbolic product $\Omega^{\frac{1}{3}ng} . S^g_{\xi\eta\zeta}$ is thereby expanded into a sum of terms, each of which is a product of $\frac{1}{3} . n . g$ determinants. The number of such terms is the number of permutations of the $n . g$ linear factors of $S^g_{\xi\eta\zeta}$. But every such term will vanish identically if the constituents of two rows in any one Ω have acted upon like factors of $S^g_{\xi\eta\zeta}$; *e. g.* upon the two factors:

$$\left(\bar{a}_\xi \frac{\partial}{\partial a_1} + \bar{a}_\eta \frac{\partial}{\partial a_2} + \bar{a}_\zeta \frac{\partial}{\partial a_3}\right)^2 .$$

There will remain therefore only terms composed of symbolic factors of the following type:

$$\begin{vmatrix} \bar{a}_1 \dfrac{\partial}{\partial a_1} & \bar{a}_2 \dfrac{\partial}{\partial a_1} & \bar{a}_3 \dfrac{\partial}{\partial a_1} \\ \bar{b}_1 \dfrac{\partial}{\partial b_2} & \bar{b}_2 \dfrac{\partial}{\partial b_2} & \bar{b}_3 \dfrac{\partial}{\partial b_2} \\ \bar{c}_1 \dfrac{\partial}{\partial c_3} & \bar{c}_2 \dfrac{\partial}{\partial c_3} & \bar{c}_3 \dfrac{\partial}{\partial c_3} \end{vmatrix} \equiv (\bar{a}\bar{b}\bar{c}) \frac{\partial^3}{\partial a_1 . \partial b_2 . \partial c_3} .$$

Indicating a term composed of $\frac{1}{3}ng$ such factors by $\prod\limits_{(\frac{1}{3}ng)} \left[(\bar{a}\bar{b}\bar{c}) \frac{\partial^3}{\partial a_1 . \partial b_2 . \partial c_3} \right]$, we may express the combined operation thus:

$$\Omega^{\frac{1}{3}ng} . S^g_{\xi\eta\zeta} . (A_{IKL}) = \sum \prod_{(\frac{1}{3}ng)} \left[(\bar{a}\bar{b}\bar{c}) \frac{\partial^3}{\partial a_1 . \partial b_2 . \partial c_3} \right] (A_{IKL}), \qquad (13)$$

where the summation extends over all possible arrangements of the symbols in each term of (A_{IKL}) in triplets such that each triplet consists of three different letters with three different indices (as $a_1 b_2 c_3$). Disregarding here the not uninteresting question as to the number of such possible arrangements, its maximum, minimum, etc., we have to notice only the formal constitution of the result. *Every term in the result is a product of* $\frac{1}{3}ng$ *factors of the type* $(\bar{a}\bar{b}\bar{c})$, *and is therefore per se invariant.* It remains finally to apply the operator $R^g(\bar{a}, a)$ whose effect is evidently *to replace each elementary determinant* $(\bar{a}\bar{b}\bar{c})$ *by* (abc), *so that the aggregate of terms becomes an invariant of* $f(x_1, x_2, x_3)$. While it would require arbitrary conventions to enable us to write the combination of the three operators in analytical symbols,—*e. g.*

$$\sum \prod_{(\frac{1}{3}ng)} [(abc)] \prod_{(\frac{1}{3}ng)} \left[\frac{\partial^3}{\partial a_1 \, . \, \partial b_2 \, . \, \partial c_3}\right],$$

yet it is easy to recapitulate in verbal form the entire process as analyzed above.

Hilbert's method for deriving an invariant of the groundform $f(x_1, x_2, x_3)$ *or* a_x^n, *from a homogeneous entire function having the degree g in the coefficients and having its three weights each equal to* $\frac{1}{3} \, . \, n \, . \, g$ *is equivalent to the following. Write each coefficient of the groundform as a symbolic product of n factors according to the Clebsch-Aronhold notation, using g different sets of n similar letters. Separate in all possible ways in every term of the function the* $n \, . \, g$ *factors into* $\frac{1}{3}ng$ *triplets, each triplet containing three different literal symbols and three subscript indices* 1, 2, 3. *Replace in each such arrangement every triplet* $a_1 b_2 c_3$ *by a corresponding three-rowed determinant* (abc). *The aggregate of resulting products is by virtue of its symbolic structure an invariant.*[*] *Q. e. d.*

This statement of the method is extended without difficulty to the derivation of an invariant from a function of the coefficients of several groundforms. Such a function is required to be homogeneous in the coefficients of each groundform separately; while in computing its three weights, which must be equal, subscript indices from all coefficients indiscriminately are to be summed up. The only formal feature in which the resulting invariant can differ from a product of invariants of single forms will be the occurrence of such symbolic factors as (aba'), $(aa'a'')$, etc.; where symbols a', a'' refer to additional groundforms $a_x'^{n_1}$, $a_x''^{n_2}$, different from the first, $a_x^n = b_x^n$. The additional groundforms may include one or more linear forms, whose coefficients we are free to regard as variables contragredient to the x_1, x_2, x_3. Simultaneous invariants involving those coefficients are usually termed *contravariants;* these demand no special treatment here. *Covariants* might be dismissed as equivalent to contravariants containing at least two sets of variables, but for convenience of application I shall state explicitly the process of deriving a covariant from a suitable function.

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[*] A single, non-symbolic operation for effecting such derivation is given by Prof. W. E. Story in the Proceedings of the London Math. Soc. for 1891-2. The operator consists namely of a product of two series, infinite in form, but having only a finite number of terms applicable to any given case; each term denoting a particular succession of elementary substitutions.

§2.—*Modified Method for Covariants, with Corollary.*

By the weight of a covariant we usually understand the exponent of that power of the modulus, by which the covariant is multiplied when its variables are transformed linearly. To be consistent, we must define the weights of a single term in the covariant in the following way. Let B_{IKL}^{Θ} denote a term of degree g in the coefficients of a groundform a_x^n, and of order Θ in several sets of variables taken together:

$$\left.\begin{aligned} B_{IKL}^{\Theta} &= a_{ikl} \,.\, a_{i'k'l'} \,.\, a_{i''k''l''} \,.\,.\,.\,.\, x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \,.\, x_1'^{\alpha'} x_2'^{\beta'} x_3'^{\gamma'} \,.\,.\,.\,. \\ \Sigma(i) + \Sigma(k) + \Sigma(l) &= ng, \\ \Sigma(\alpha) + \Sigma(\beta) + \Sigma(\gamma) &= \Theta. \end{aligned}\right\} \qquad (14)$$

Let I, K, L denote the three weights; they must be equal to the expressions

$$\Sigma(i) - \Sigma(\alpha),\ \Sigma(k) - \Sigma(\beta),\ \Sigma(l) - \Sigma(\gamma) \text{ respectively.}$$

Hence $$I + K + L = ng - \Theta.$$

Moreover the three must be equal, and accordingly:

$$I = K = L = \tfrac{1}{3}(ng - \Theta) \text{ must be an integer.} \qquad (15)$$

When a polynomial is given, every term of which fulfils these conditions with the same g and Θ,—

$$(B_{IKL}^{\Theta}) = \Sigma B_{IKL}^{\Theta}; \qquad (16)$$

then only the first step of Hilbert's process needs supplementing, in order to derive from (B_{IKL}^{Θ}) a complete covariant.

Solving the linear substitution (6), we find expressions for the x's in terms of the y's:

$$\left\{\begin{aligned} (\xi\eta\zeta)x_1 &= (y\eta\zeta) \\ (\xi\eta\zeta)x_2 &= (y\zeta\xi) \\ (\xi\eta\zeta)x_3 &= (y\xi\eta) \end{aligned}\right\} \qquad (17)$$

The first step (p. 4) will now be *to substitute in the polynomial* (B_{IKL}^{Θ}) *for the coefficients* a_{ikl} *their equivalents in terms of the* a_{ikl}, *for the several variables* (x_1, x_2, x_3; x_1', x_2', x_3'; . . . .) *their equivalents in terms of transformed variables* (y_1, y_2, y_3; y_1', y_2', y_3'; . . . .); *and to multiply the result by such a power of the modulus of substitution*, $(\xi\eta\zeta)$, *as will make the product an entire function of the nine parameters* ξ, η, ζ. The multiplier is obviously not of higher degree than the Θth; and as a superfluous factor here means merely a numerical factor in the final result, we may always employ

$$(\xi\eta\zeta)^{\Theta}.$$

After this first step the result may be symbolized by

$$(\xi\eta\zeta)^{\Theta}(\overline{B}^{\Theta}_{IKL}).$$

The second step is the same as in §1, save that Ω has an exponent slightly altered. Its effect is written symbolically:

$$\Omega^{\frac{1}{3}(ng+2\Theta)}\,.\,(\xi\eta\zeta)^{\Theta}\,.\,(\overline{B}^{\Theta}_{IKL}) = J(\bar{a}_{ikl};\ y). \tag{18}$$

Lastly there is to be included in the third step the replacement of variables (y) by variables (x); $J(a_{ikl};\ x)$ is then a covariant, and the process by which it is derived from (B^{Θ}_{IKL}) is perfectly determinate.

This process enables me to verify neatly an important theorem, which may be given here as a corollary. Suppose that $(B^{\Theta}_{IKL}) = F(a_{ikl};\ x,\ x')$ is a function of coefficients of a single groundform: a^n_x, and of only two sets of variables: $(x_1,\ x_2,\ x_3;\ x'_1,\ x'_2,\ x'_3)$. Apply the operation described as the first step, and suppose further that an identity subsists as follows:

$$(\xi\eta\zeta)^{\Theta}(B^{\Theta}_{IKL}) \equiv \overline{F}(\bar{a}_{ikl};\ y,\ y') \equiv M\,.\,F(\bar{a}_{ikl};\ y,\ y') + G_1\,.\,\bar{a}^n_y + G_2\,.\,\bar{a}^n_{y'} \tag{19}$$

(see formulae 9); G_1 and G_2 denoting entire functions of both coefficients and variables, M a function of parameters ξ, η, ζ only. By the second operation we obtain evidently:

$$J(\bar{a}_{ikl};\ y,\ y') \equiv F(\bar{a}_{ikl};\ y,\ y')\,.\,(\Omega^{\frac{1}{3}(ng+2\Theta)}M) + \bar{a}^n_y\,.\,(\Omega^{\frac{1}{3}(ng+2\Theta)}G_1) + \bar{a}^n_{y'}\,.\,(\Omega^{\frac{1}{3}(ng+2\Theta)}G_2).$$

Hence the final result is a relation of the form:

$$J(a_{ikl};\ x,\ x') \doteq M'\,.\,F(a_{ikl};\ x,\ x') + G'_1\,.\,a^n_x + G'_2\,.\,a^n_{x'}, \tag{20}$$

in which M' is a constant, G'_1 and G'_2 entire functions of coefficients and variables. This is stated as a proposition thus, adding obvious extensions of the above proof:

COROLLARY: *If an entire homogeneous function of the coefficients of several groundforms and of several cogredient sets of variables, of suitable weights, have the property that every linear transformation of the variables reproduces the function,* MODULIS *the given groundform; then it is possible to add to the function certain determinate multiples of the groundforms (written in the variables of each set separately), such that the sum is a covariant, i. e. reproduces itself with no additive terms upon every linear transformation of the variables.* Applications of this theorem arise in the study of point-systems upon algebraic curves.

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